Solutions to CRMO-2003

1. Let ABC be a triangle in which $AB = AC$ and $\angle CAB = 90^\circ$. Suppose M and N are points on the hypotenuse BC such that $BM^2 + CN^2 = MN^2$. Prove that $\angle MAN = 45^\circ$.

   Solution:

   Draw CP perpendicular to CB and BQ perpendicular to CB such that $CP = BM$, $BQ = CN$. Join PA, PM, PN, QA, QM, QN. (See Fig. 1.)

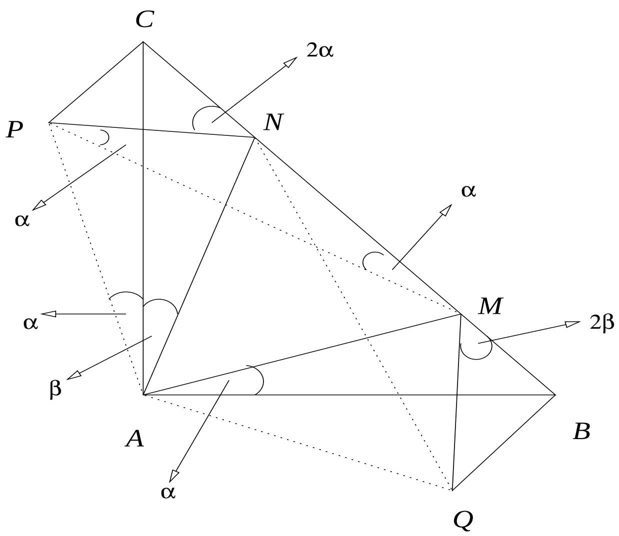

Fig. 1.

   In triangles CPA and BMA, we have $\angle PCA = 45^\circ = \angle MBA$; $PC = MB$, $CA = BA$. So $\triangle CPA \equiv \triangle BMA$. Hence $\angle PAC = \angle BAM = \alpha$, say. Consequently, $\angle MAP = \angle BAC = 90^\circ$, whence $PAMC$ is a cyclic quadrilateral. Therefore $\angle PMC = \angle PAC = \alpha$. Again $PN^2 = PC^2 + CN^2 = BM^2 + CN^2 = MN^2$. So $PN = MN$, giving $\angle NPM = \angle NMP = \alpha$, in $\triangle PMN$. Hence $\angle PNC = 2\alpha$. Likewise $\angle QMB = 2\beta$, where $\beta = \angle CAN$. Also $\triangle NCP \equiv \triangle QBM$, as $CP = BM, NC = BQ$ and $\angle NCP = 90^\circ = \angle QBM$. Therefore, $\angle CPN = \angle BMQ = 2\beta$, whence $2\alpha + 2\beta = 90^\circ$; $\alpha + \beta = 45^\circ$; finally $\angle MAN = 90^\circ - (\alpha + \beta) = 45^\circ$.

   Aliter: Let $AB = AC = a$, so that $BC = \sqrt{2}a$; and $\angle MAB = \alpha, \angle CAN = \beta$.(See Fig. 2.)

   By the Sine Law, we have from $\triangle ABM$ that

$$\frac{BM}{\sin\alpha} = \frac{AB}{\sin(\alpha + 45^\circ)}.$$

So $BM = \dfrac{a\sqrt{2}\sin\alpha}{\cos\alpha + \sin\alpha} = \dfrac{a\sqrt{2}u}{1+u}$, where $u = \tan\alpha$.

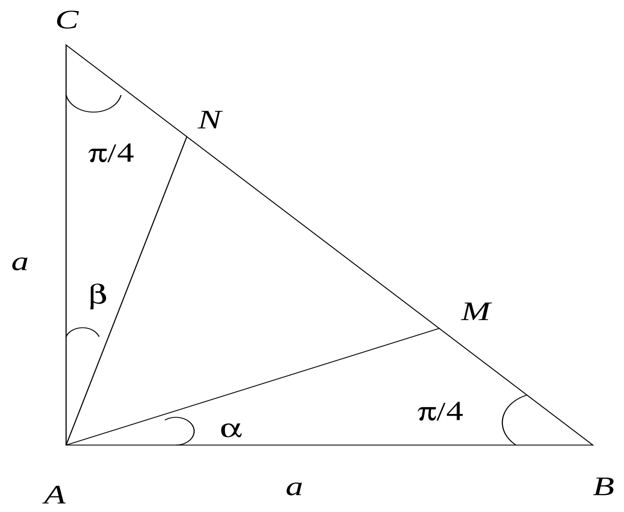

Fig. 2.

Similarly $CN = \dfrac{a\sqrt{2}v}{1+v}$, where $v = \tan\beta$. But

$$\begin{aligned} BM^2 + CN^2 &= MN^2 = (BC - MB - NC)^2 \\ &= BC^2 + BM^2 + CN^2 \\ &\qquad - 2BC \cdot MB - 2BC \cdot NC + MB \cdot NC. \end{aligned}$$

So

$$BC^2 - 2BC \cdot MB - 2BC \cdot NC + 2MB \cdot NC = 0.$$

This reduces to

$$2a^2 - 2\sqrt{2}a\frac{a\sqrt{2}u}{1+u} - 2\sqrt{2}a\frac{a\sqrt{2}v}{1+v} + \frac{4a^2uv}{(1+u)(1+v)} = 0.$$

Multiplying by $(1+u)(1+v)/2a^2$, we obtain

$$(1+u)(1+v) - 2u(1+v) - 2v(1+u) + 2uv = 0.$$

Simplification gives $1 - u - v - uv = 0$. So

$$\tan(\alpha + \beta) = \frac{u+v}{1-uv} = 1.$$

This gives $\alpha + \beta = 45°$,whence $\angle MAN = 45°$, as well.

2. If n is an integer greater than 7, prove that $\binom{n}{7} - \left[\frac{n}{7}\right]$ is divisible by 7. $\Big[$Here $\binom{n}{7}$ denotes the number of ways of choosing 7 objects from among n objects; also, for any real number x, $[x]$ denotes the greatest integer not exceeding x.$\Big]$

Solution: We have

$$\binom{n}{7} = \frac{n(n-1)(n-2)\dots(n-6)}{7!}.$$

In the numerator, there is a factor divisible by 7, and the other six factors leave the remainders 1,2,3,4,5,6 in some order when divided by 7.

Hence the numerator may be written as

$$7k \cdot (7k_1 + 1) \cdot (7k_2 + 2) \cdots (7k_6 + 6).$$

Also we conclude that $\left[\frac{n}{p}\right] = k$, as in the set $\{n, n-1, \dots n-6\}$, $7k$ is the only number which is a multiple of 7. If the given number is called Q, then

$$\begin{aligned} Q &= 7k \cdot \frac{(7k_1+1)(7k_2+2)\dots(7k_6+6)}{7!} - k \\ &= k\left[\frac{(7k_1+1)\dots(7k_6+6) - 6!}{6!}\right] \\ &= \frac{k[7t + 6! - 6!]}{6!} \\ &= \frac{7tk}{6!}. \end{aligned}$$

We know that Q is an integer, and so $6!$ divides $7tk$. Since $\gcd(7, 6!) = 1$, even after cancellation there is a factor of 7 still left in the numerator. Hence 7 divides Q, as desired.

3. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Prove that among the three numbers $a - ab$, $b - bc$, $c - ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$.

Solution: By AM-GM inequality, we have

$$a(1-a) \le \left(\frac{a+1-a}{2}\right)^2 = \frac{1}{4}.$$

Similarly we also have

$$b(1-b) \le \frac{1}{4} \quad \text{and} \quad c(1-c) \le \frac{1}{4}.$$

Multiplying these we obtain

$$abc(1-a)(1-b)(1-c) \le \frac{1}{4^3}.$$

We may rewrite this in the form

$$a(1-b)\cdot b(1-c)\cdot c(1-a)\leq \frac{1}{4^3}.$$

Hence one factor at least $\big($among $a(1-b), b(1-c), c(1-a)\big)$ has to be less than or equal to $\frac{1}{4}$; otherwise **lhs** would exceed $\frac{1}{4^3}$.

Again consider the sum $a(1-b)+b(1-c)+c(1-a)$. This is equal to $a+b+c-ab-bc-ca$. We observe that

$$3\big(ab+bc+ca\big)\leq \big(a+b+c\big)^2,$$

which, in fact, is equivalent to $(a-b)^2+(b-c)^2+(c-a)^2\geq 0$. This leads to the inequality

$$a+b+c-ab-bc-ca\geq (a+b+c)-\frac{1}{3}(a+b+c)^2 = 1-\frac{1}{3}=\frac{2}{3}.$$

Hence one summand at least $\big($ among $a(1-b), b(1-c), c(1-a)\big)$ has to be greater than or equal to $\frac{2}{9}$; $\big($otherwise **lhs** would be less than $\frac{2}{3}$.$\big)$

4. Find the number of ordered triples (x, y, z) of nonnegative integers satisfying the conditions:

   (i) $x\leq y\leq z$;

   (ii) $x+y+z\leq 100$.

   Solution: We count by brute force considering the cases $x=0, x=1, \ldots, x=33$. Observe that the least value x can take is zero, and its largest value is 33.

   <u>x = 0</u> If $y=0$, then $z\in\big\{0,1,2,\ldots,100\big\}$; if y=1, then $z\in\big\{1,2,\ldots,99\big\}$; if $y=2$, then $z\in\big\{2,3,\ldots,98\big\}$; and so on. Finally if $y=50$, then $z\in\{50\}$. Thus there are altogether $101+99+97+\cdots+1=51^2$ possibilities.

   <u>x = 1.</u> Observe that $y\geq 1$. If $y=1$, then $z\in\big\{1,2,\ldots,98\big\}$; if $y=2$, then $z\in\big\{2,3,\ldots,97\big\}$; if $y=3$, then $z\in\big\{3,4,\ldots,96\big\}$; and so on. Finally if $y=49$, then $z\in\big\{49,50\big\}$. Thus there are altogether $98+96+94+\cdots+2=49\cdot 50$ possibilities.

   <u>General case.</u> Let x be even, say, $x=2k$, $0\leq k\leq 16$. If $y=2k$, then $z\in\big\{2k, 2k+1,\ldots,100-4k\big\}$; if $y=2k+1$, then $z\in\big\{2k+1,2k+2,\ldots,99-4k\big\}$; if $y=2k+2$, then $z\in\big\{2k+2,2k+3,\ldots,99-4k\big\}$; and so on.

   Finally, if $y=50-k$, then $z\in\big\{50-k\big\}$. There are altogether

$$(101-6k)+(99-6k)+(97-6k)+\cdots+1=(51-3k)^2$$

   possibilities.

Let x be odd, say, $x = 2k+1$, $0 \le k \le 16$. If $y = 2k+1$, then $z \in \{2k+1, 2k+2, \ldots, 98-4k\}$; if $y = 2k+2$, then $z \in \{2k+2, 2k+3, \ldots, 97-4k\}$; if $y = 2k+3$, then $z \in \{2k+3, 2k+4, \ldots, 96-4k\}$; and so on.

Finally, if $y = 49-k$, then $z \in \{49-k, 50-k\}$. There are altogether

$$(98-6k) + (96-6k) + (94-6k) + \cdots + 2 = (49-3k)(50-3k)$$

possibilities.

The <u>last two cases</u> would be as follows:

<u>**x = 32**</u>: if $y = 32$, then $z \in \{32, 33, 34, 35, 36\}$; if $y = 33$, then $z \in \{33, 34, 35\}$; if $y = 34$, then $z \in \{34\}$; altogether $5+3+1 = 9 = 3^2$ possibilities.

<u>**x = 33**</u>: if $y = 33$, then $z \in \{33, 34\}$; only 2=1.2 possibilities.

Thus the total number of triples, say T, is given by,

$$T = \sum_{k=0}^{16} (51-3k)^2 + \sum_{k=0}^{16} (49-3k)(50-3k).$$

Writing this in the reverse order, we obtain

$$\begin{aligned} T &= \sum_{k=1}^{17} (3k)^2 + \sum_{k=0}^{17} (3k-2)(3k-1) \\ &= 18 \sum_{k=1}^{17} k^2 - 9 \sum_{k=1}^{17} k + 34 \\ &= 18 \left(\frac{17 \cdot 18 \cdot 35}{6} \right) - 9 \left(\frac{17 \cdot 18}{2} \right) + 34 \\ &= 30,787. \end{aligned}$$

Thus the answer is 30787.

<u>Aliter</u>

It is known that the number of ways in which a given positive integer $n \ge 3$ can be expressed as a sum of three positive integers x, y, z (that is, $x + y + z = n$), subject to the condition $x \le y \le z$ is $\left\{ \frac{n^2}{12} \right\}$, where $\{a\}$ represents the integer closest to a. If zero values are allowed for x, y, z then the corresponding count is $\left\{ \frac{(n+3)^2}{12} \right\}$, where now $n \ge 0$.

Since in our problem $n = x + y + z \in \{0, 1, 2, \ldots, 100\}$, the desired answer is

$$\sum_{n=0}^{100} \left\{ \frac{(n+3)^2}{12} \right\}.$$

For $n = 0, 1, 2, 3, \ldots, 11$, the corrections for $\{\}$ to get the nearest integers are

$$\frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}, \frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}.$$

So, for 12 consecutive integer values of n, the sum of the corrections is equal to

$$\left(\frac{3-4-1-0-1-4-3}{12}\right) \times 2 = \frac{-7}{6}.$$

Since $\frac{101}{12} = 8 + \frac{5}{12}$, there are 8 sets of 12 consecutive integers in $\{3,4,5,\ \ldots\ ,103\ \}$ with 99,100,101,102,103 still remaining. Hence the total correction is

$$\left(\frac{-7}{6}\right) \times 8 + \frac{3-4-1-0-1}{12} = \frac{-28}{3} - \frac{1}{4} = \frac{-115}{12}.$$

So the desired number T of triples (x, y, z) is equal to

$$\begin{aligned} T &= \sum_{n=0}^{100} \frac{(n+3)^2}{12} - \frac{115}{12} \\ &= \frac{\left(1^2 + 2^2 + 3^2 + \cdots + 103^2\right) - \left(1^2 + 2^2\right)}{12} - \frac{115}{12} \\ &= \frac{103 \cdot 104 \cdot 207}{6 \cdot 12} - \frac{5}{12} - \frac{115}{12} \\ &= 30787. \end{aligned}$$

5. Suppose P is an interior point of a triangle ABC such that the ratios

$$\frac{d(A, BC)}{d(P, BC)}, \quad \frac{d(B, CA)}{d(P, CA)}, \quad \frac{d(C, AB)}{d(P, AB)}$$

are all equal. Find the common value of these ratios. [Here $d(X, YZ)$ denotes the perpendicular distance from a point X to the *line* YZ.]

Solution: Let AP, BP, CP when extended, meet the sides BC, CA, AB in D, E, F respectively. Draw AK, PL perpendicular to BC with K, L on BC.(See Fig. 3.)

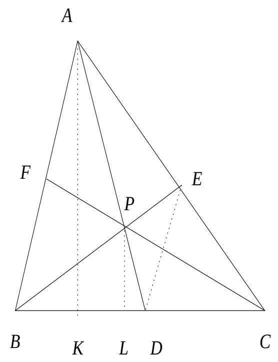

Fig. 3.

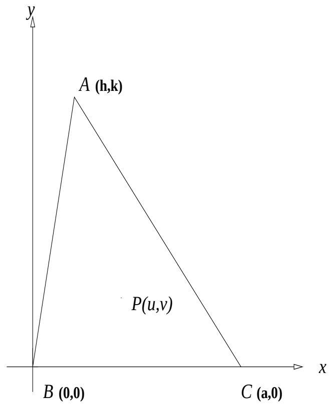

Fig. 4.

Now

$$\frac{d(A,BC)}{d(P,BC)} = \frac{AK}{PL} = \frac{AD}{PD}.$$

Similarly,

$$\frac{d(B,CA)}{d(P,CA)} = \frac{BE}{PE} \quad \text{and} \frac{d(C,AB)}{d(P,AB)} = \frac{CF}{PF}.$$

So, we obtain

$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF}, \quad \text{and hence} \quad \frac{AP}{PD} = \frac{BP}{PE} = \frac{CP}{PF}.$$

From $\dfrac{AP}{PD} = \dfrac{BP}{PE}$ and $\angle APB = \angle DPE$, it follows that triangles APB and DPE are similar. So $\angle ABP = \angle DEP$ and hence AB is parallel to DE.

Similarly, BC is parallel to EF and CA is parallel to DF. Using these we obtain

$$\frac{BD}{DC} = \frac{AE}{EC} = \frac{AF}{FB} = \frac{DC}{BD},$$

whence $BD^2 = CD^2$ or which is same as $BD = CD$. Thus D is the midpoint of BC. Similarly E, F are the midpoints of CA and AB respectively.

We infer that AD, BE, CF are indeed the medians of the triangle ABC and hence P is the centroid of the triangle. So

$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF} = 3,$$

and consequently each of the given ratios is also equal to 3.

<u>Aliter</u>

Let ABC, the given triangle be placed in the xy-plane so that $B = (0,0), C = (a,0)$ (on the x- axis). (See Fig. 4.)

Let $A = (h,k)$ and $P = (u,v)$. Clearly $d(A,BC) = k$ and $d(P,BC) = v$, so that

$$\frac{d(A,BC)}{d(P,BC)} = \frac{k}{v}.$$

The equation to CA is $kx - (h-a)y - ka = 0$. So

$$\begin{aligned}\frac{d(B,CA)}{d(P,CA)} &= \frac{-ka}{\sqrt{k^2+(h-a)^2}} \Big/ \frac{(ku-(h-a)v-ka)}{\sqrt{k^2+(h-a)^2}} \\ &= \frac{-ka}{ku-(h-a)v-ka}.\end{aligned}$$

Again the equation to AB is $kx - hy = 0$. Therefore

$$\begin{aligned}\frac{d(C,AB)}{d(P,AB)} &= \frac{ka}{\sqrt{h^2+k^2}} \Big/ \frac{(ku-hv)}{\sqrt{h^2+k^2}} \\ &= \frac{ka}{ku-hv}.\end{aligned}$$

From the equality of these ratios, we get

$$\frac{k}{v} = \frac{-ka}{ku-(h-a)v-ka} = \frac{ka}{ku-hv}.$$

The equality of the first and third ratios gives $ku-(h+a)v = 0$. Similarly the equality of second and third ratios gives $2ku - (2h-a)v = ka$. Solving for u and v, we get

$$u = \frac{h+a}{3}, \quad v = \frac{k}{3}.$$

Thus P is the centroid of the triangle and each of the ratios is equal to $\frac{k}{v} = 3$.

6. Find all real numbers a for which the equation

$$x^2 + (a-2)x + 1 = 3|x|$$

has exactly three distinct real solutions in x.

Solution: If $x \geq 0$, then the given equation assumes the form,

$$x^2 + (a-5)x + 1 = 0. \qquad \cdots(1)$$

If $x < 0$, then it takes the form

$$x^2 + (a+1)x + 1 = 0. \qquad \cdots(2)$$

For these two equations to have exactly three distinct real solutions we should have

(I) either $(a-5)^2 > 4$ and $(a+1)^2 = 4$;

(II) or $(a-5)^2 = 4$ and $(a+1)^2 > 4$.

Case (I) From $(a+1)^2 = 4$, we have $a = 1$ or -3. But only $a = 1$ satisfies $(a-5)^2 > 4$. Thus $a = 1$. Also when $a = 1$, equation (1) has solutions $x = 2 + \sqrt{3}$; and (2) has solutions $x = -1, -1$. As $2 \pm \sqrt{3} > 0$ and $-1 < 0$, we see that $a = 1$ is indeed a solution.

Case (II) From $(a-5)^2 = 4$, we have a=3 or 7. Both these values of a satisfy the inequality $(a+1)^2 > 4$. When $a = 3$, equation (1) has solutions $x = 1, 1$ and (2) has the solutions $x = -2 \pm \sqrt{3}$. As $1 > 0$ and $-2 \pm \sqrt{3} < 0$, we see that $a = 3$ is in fact a solution.

When $a = 7$, equation (1) has solutions $x = -1, -1$, which are negative contradicting $x \geq 0$.

Thus $a = 1, a = 3$ are the two desired values.

7. Consider the set $X = \{1, 2, 3, \dots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

   (a) $A \cup B = X$;

   (b) prod(A) is divisible by prod(B), where for any finite set of numbers C, prod(C) denotes the product of all numbers in C ;

   (c) the quotient prod(A)/prod(B) is as small as possible.

Solution: The prime factors of the numbers in set $\{$1,2,3, ..., 9,10$\}$ are 2,3,5,7. Also only $7 \in X$ has the prime factor 7. Hence it cannot appear in B. For otherwise, 7 in the denominator would not get canceled. Thus $7 \in A$.

Hence

$$\text{prod}(A)/\text{prod}(B) \geq 7.$$

The numbers having prime factor 3 are 3,6,9. So 3 and 6 should belong to one of A and B, and 9 belongs to the other. We may take $3, 6 \in A$, $9 \in B$.

Also 5 divides 5 and 10. We take $5 \in A$, $10 \in B$. Finally we take $1, 2, 4 \in A$, $8 \in B$. Thus

$$A = \{1, 2, 3, 4, 5, 6, 7\}, \quad B = \{8, 9, 10\},$$

so that

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{8 \cdot 9 \cdot 10} = 7.$$

Thus 7 is the minimum value of $\dfrac{\text{prod}(A)}{\text{prod}(B)}$. There are other possibilities for A and B: e.g., 1 may belong to either A or B. We may take $A = \{3, 5, 6, 7, 8\}$, $B = \{1, 2, 4, 9, 10\}$.